Localized buckling of a floating Elastica

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We study the buckling of a 2D Elastica floating on a bath of dense fluid, subjected to axial compression. The sinusoidal pattern predicted by the analysis of linear stability is shown to become localized above the buckling threshold. A non-linear amplitude equation is derived for the envelope of the pattern. These results provide a simple interpretation to the wrinkle-to-fold transition reported by Pocivavsek et al. (Science, 2008). An analogy with the classical problem of the localized buckling of a strut on an non-linear elastic foundation is presented.

I. INTRODUCTION

The buckling of an elastic rod resting on an elastic foundation is a classical problem in structural engineering. While the critical load and wavelength are set by the size of the system in the case of Euler's free-standing Elastica, an intrinsic length scale appears in the presence of a foundation. This scale determines a critical load and wavelength that are independent of the size of the system, provided it is large enough.

This classical problem and its variants have received an upsurge of interest recently, see Ref. [1] for a review. A stiff film, skin or filament buckling on a soft foundation define simple systems displaying rich wrinkling behaviors governed by well-controlled, geometrical nonlinearities. In gels, swelling induces biaxial residual stress along a skin near the surface, whose buckling and post-buckling behavior has been investigated [2, 3]. The nonlinear phenomenon of period doubling has been observed the wrinkling of films bound to an elastomer [4]. Nonlinear selection of 2D buckling patterns in a thin, stiff elastic plate on top of a soft foundation has been described recently [5–7]. The related geometry of a thin, stiff shell around a soft spheroidal core has been investigated numerically in connection with the morphogenesis of fruits and vegetables [8].

An extreme case of a soft foundation is that of a film floating on a fluid. A striking, self-similar wrinkling pattern has been observed in this case [9]. In the present paper, we analyze recent observations of the buckling of a thin polymer sheet resting on the surface of water by Pocivavsek [10]. The film is compressed laterally by clamping its lateral edges. Immediately above threshold, sinusoidal wrinkles are formed. They spread over the entire length of the film and are well described by a linear stability analysis. When the sheet is further compressed, the authors report a transition to sharp, localized folds. These observations were based on experiments, and reproduced in numerical simulations. Here, we show that this wrinkle-to-fold transition is a particular example of the phenomenon of localized buckling.

In a recent paper, Diamant and Witten [11] showed that the experimental and numerical results of Pocivavsek, including the wrinkle-to-fold transition, are well explained by postulating a locally sinusoidal buckling, with a slowly varying envelope given by a hyperbolic secant function. They found that this particular buckling profile has indeed a lower energy than the uniform sinusoidal pattern sufficiently close to threshold. They tried other profiles for the envelope but did not find any that would make the energy lower or would agree better with the experiments. Here, we show that the slow modulation of the buckling amplitude and the wrinkle-to-fold transitions are related to a phenomenon known as localized buckling, that the optimal envelope can be derived systematically using an amplitude equation, and that the hyperbolic secant profile that they proposed is indeed optimal.

Amplitude equation were initially introduced in the context of convection phenomena in fluid mechanics, to explain the organization of rolls in post-critical Rayleigh-Bénard convection [12, 13]; see Ref. [14] for a review. In the field of mechanical engineering, amplitude equations were used as part of a general effort to characterize the post-buckled behavior of structures [15]. A related two-scale expansion was introduced by Amazigo [16], who pointed out localization of buckling patterns induced by imperfections for a beam resting on a non-linear elastic foundation. A general discussion on this problem can be found in Ref. [17]. An interesting interpretation is developed by Tvergaard [18]: using an elegant and generic model, they interpret localized buckling as the progressive invasion of a phase representing uniform buckles, by a phase representing localized buckles. A number of extensions have been considered, such as the case of a spatially inhomogeneous foundation [19, 20], and dynamic and 3D effects [21, 22]. The buckling of a twisted elastic rod into a helical pattern is a well-known illustration of the phenomenon of localization. The exact non-linear solution of Coyne [23] resembles an amplitude equation but is exact to all orders. Its stability been investigated analytically and theoretically [24].

II. PROBLEM FORMULATION

We consider an inextensible elastic filament of length L floating on a bath of dense fluid in a 2D geometry. The main unknown is the profile $h(s)$ of the filament, parameterized by the arc-length s; see Figure 1. Following

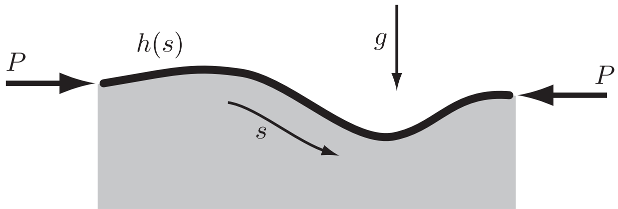

FIG. 1. Buckling of an Elastica floating on a bath of fluid. The profile is expressed by the function $h(s)$ yielding the deflection as a function of arc-length.

Reference [11], we consider the energy in rescaled form

$$E = \int_{-L/2}^{L/2} e \, \mathrm{d}s, \tag{1a}$$

with density per arclength

$$e = \frac{1}{2} \frac{h''^2}{1 - h'^2} - P \left(1 - \left(1 - h'^2 \right)^{\frac{1}{2}} \right) + U(h, h'). \tag{1b}$$

The first term is a bending energy expressed in units such that the bending modulus is unity. The second term is the work of the external horizontal compression force P; the coefficient of P in equation (1b) yields upon integration the horizontal separation of the endpoints of the filament. The last term $U(h, h')$ is the potential energy of the foundation. In the floating case, this is the energy of a column of of fluid which is pushed downwards or sucked upwards by a height $h(s)$ away from its natural level:

$$U(h, h') = U_{\mathrm{f}}(h, h') = \frac{h^2}{2} \left(1 - h'^2 \right)^{\frac{1}{2}}. \tag{2}$$

In parallel with the floating case, we shall consider the classical case of a non-linear elastic foundation, known as a Winkler foundation:

$$U(h, h') = U_{\mathrm{w}}(h) = \frac{1}{2} h^2 + \frac{\nu_{\mathrm{w}}}{24} h^4, \tag{3}$$

where the parameter ν_{w} measures the amount of non-linearity. The case of an elastic foundation, when the foundation energy does not depend on the derivative h', has been discussed at length in the literature. One of the contribution of the present paper is to extend the theory of localized buckling, classically based on equation (3), to a more general foundation energy of the form $U(h, h')$.

With the aim to highlight the analogy between the two cases, we will avoid to make any use of the particular forms of the fluid or elastic potentials (2) and (3). We shall simply make use of the following properties:

$$U(0, 0) = 0 \tag{4a}$$

$$U_{,hh'}(0, 0) = 0 \tag{4b}$$

$$U_{,h^2}(0, 0) = 1 \tag{4c}$$

$$U_{,h'^2}(0, 0) = 0. \tag{4d}$$

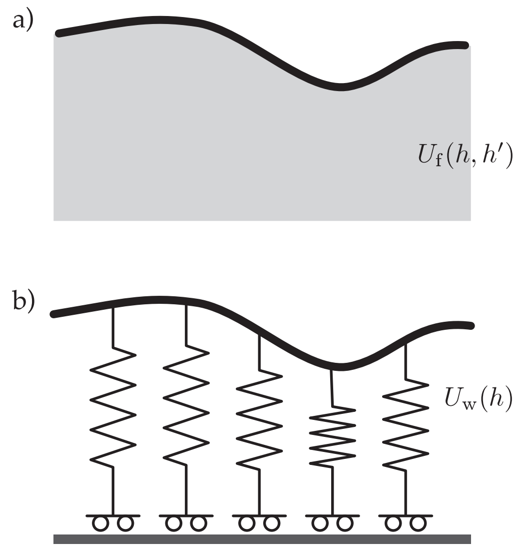

FIG. 2. We are mainly interested in the case (a) of a floating Elastica, represented by a potential $U_{\mathrm{f}}(h, h')$. Our derivation of the amplitude equation extends the classical case (b) of a beam resting on a non-linear elastic foundation (Winkler foundation). In the latter case, the potential $U_{\mathrm{w}}(h)$ does not depend on the local slope h'.

These equalities are satisfied by both types of foundations (2) and (3), as can be checked easily. Note that (i) equation (4a) can always be satisfied by redefining the zero of energy, (ii) equation (4b) follows from the symmetries $h \leftarrow (-h)$ and $h' \leftarrow (-h')$, and (iii) equation (4c) yields the linearized stiffness of the foundation which can indeed be set to 1 by appropriate rescaling.

Let us define an anharmonicity parameter for the foundation, denoted $\nu(U)$, whose importance will become clear later:

$$\nu(U) = U_{,h^4}(0, 0) + 2\, U_{,h^2 h'^2}(0, 0) + U_{,h'^4}(0, 0). \tag{5}$$

Evaluation of the derivatives of the fluid potential U_{f} in equation (2) yields the value

$$\nu_{\mathrm{f}} = \nu(U_{\mathrm{f}}) = -2 \tag{6}$$

for the floating case. In the case of a non-linear elastic foundation, equation (5) yields $\nu(U_{\mathrm{w}}) = \nu_{\mathrm{w}}$ showing that the notation ν_{w} in equations (3) is consistent with the definition (5).

In the following sections, we study the equilibria of the floating filament, i. e. determine profiles $h(s)$ making the energy of equation (1) stationary. We do not consider stability issues which have been discussed in related problems, see e. g. Ref. [24], and in the problem at hand [11]. As a result, we can replace the displacement control in the original experiments by a force control in the present analysis, through the load parameter P. This avoids the complication to deal with a constrained minimization problem. The experiment would yield different results with a control in force, which is unstable, but that is irrelevant as far as the equilibria are concerned.

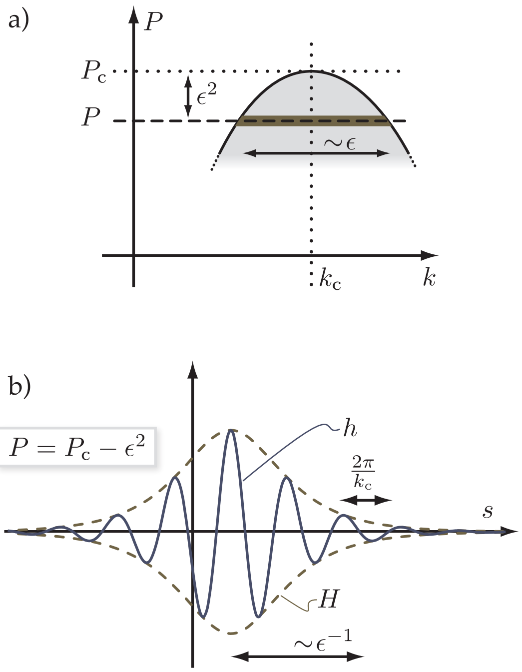

FIG. 3. Interpretation of the two-scale expansion (7a) based on the existence of a small width $\Delta k \sim \epsilon$ of unstable wavenumbers near threshold, after [12].

Nonlinearities in the response of the foundation, here quantified by the parameter $\nu(U)$, are known to cause localization of buckling, see e. g. Ref. [17]. Geometric nonlinearities in the Elastica energy (1b) are also known to produced localization [25]. Here, both effects are taken into account.

III. TWO-SCALE EXPANSION

Near the threshold for linear stability $P = P_c$, there is a width $\Delta k \sim \sqrt{|P - P_c|}$ of unstable wavenumbers, as sketched in Figure 3. By a classical argument [12], this unstable band yields by linear combination a slowly modulated sinusoidal pattern with local wavenumber k_c, the critical wavenumber. This motivates the following two-scale expansion [13, 14]

$$h(s) = \epsilon \left(H_1 \left(\frac{s}{1/\epsilon} \right) + \epsilon\, H_2 \left(\frac{s}{1/\epsilon} \right) + \cdots \right) \cos(k_c\, s). \tag{7a}$$

Here, ϵ is defined as the square-root of distance to threshold:

$$P = P_c - \epsilon^2. \tag{7b}$$

Note that the dominant term in the expansion (7a) is linear in ϵ; by equation (7b), the amplitude is therefore proportional to the square root of the distance to threshold, as usual in continuous bifurcations. In other words, the indeterminacy in the sign of $\epsilon = \pm\sqrt{P_c - P}$ corresponds to the up-down symmetry $h \leftarrow (-h)$ of the system.

In equation (7b), a minus sign is required in front of the term ϵ^2: replacing it by a plus sign would make the forthcoming construction fail. This points to the fact that buckled solutions exist only for loads less than the critical load, $P \leq P_c$, *i. e.* the load decreases above threshold. Such a bifurcation is called sub-critical. Because of this decrease of the load, a force-controlled experiment would be unstable but there is no discontinuity if the displacement is controlled instead.

Our aim is to derive the shape H_1 of the envelope at leading order near threshold, i. e. when the parameter ϵ is small:

$$\epsilon \ll 1. \tag{8}$$

For simplicity, we assume that the size of the system is infinite:

$$L \gg \frac{1}{\epsilon}. \tag{9}$$

For a discussion of the effects associated with a finite size, including the selection of the unstable wavelength, see Ref. [26–28].

IV. LINEAR STABILITY

We start with the classical analysis of linear stability, which yields the critical wavelength and load but not the amplitude. It will be complemented by a non-linear analysis in the following sections.

Inserting the two-scale expansion (7) into the density of energy (1b), and expanding to second order in ϵ, we find

$$e = \epsilon\, H_1(s\,\epsilon)\ (U_{,h}(0,0)\, \cos k_c s - U_{,h'}(0,0)\, k_c\, \sin k_c s) \cdots$$
$$+ \frac{\epsilon^2}{2}\, H_1^2(s\,\epsilon)\ \left((1 + k_c^4)\, \cos^2 k_c s - P_c\, k_c^2\, \sin^2 k_c s\right) + \mathcal{O}(\epsilon^3),$$

after using the numeric values of the derivatives of $U(h, h')$ that are known from equations (4a)–(4d). Anticipating on the result $k_c = \mathcal{O}(1)$, we note that in the limit of a long Elastica, $L\, k_c \gg 1$ and near threshold, $\epsilon \ll 1$, oscillatory terms can be averaged. This removes first-order contribution, and we find

$$\frac{E}{L} = \frac{\epsilon^2}{4}\, (1 + k_c^4 - P_c\, k_c^2) \left(\frac{1}{L} \int_{-L/2}^{L/2} {H_1}^2(s\,\epsilon)\, \mathrm{d}s \right) + \mathcal{O}(\epsilon^3) \tag{10}$$

The linear stability threshold is reached when the coefficient in front of the quadratic term cancels, $(1 + k^4 - P_c\, k^2) = 0$. This yields a buckling load $P_c^*(k)$ that depends on the wavenumber of the perturbation:

$$P_c^*(k) = \frac{1 + k^4}{k^2}.$$

The critical wavenumber k_c is the found by requiring that this critical load is an extremum, $\mathrm{d}P_c^*/\mathrm{d}k = 0$, which yields $k_c = 1$. Plugging back into equation above, we find the classical result of the linear stability analysis:

$$k_c = 1, \qquad P_c = 2. \tag{11}$$

where $P_c = P_c^*(k_c)$.

V. AMPLITUDE EQUATION

In non-conservative systems such as Rayleigh-Bénard convection rolls, amplitude equations are derived by expanding the non-linear equations of motion [14]. Here the existence of a variational structure provides a more straightforward way to derive the amplitude equation, see e. g. [25, 29]: we expand the energy functional in ϵ to the lowest order where it depends explicitly on the amplitude H_1. An amplitude equation is then obtained by minimizing of the resulting functional using the Euler-Lagrange equations.

We observe that the expansion (10) of energy to order ϵ^2 is actually zero when the critical values k_c and P_c are inserted. At next order, ϵ^3, the energy is made up of quickly oscillatory terms that all average to 0, as happened at linear order in ϵ. Therefore, we have to push the expansion of E to order ϵ^4. As we shall see, the energy not only depends on the value of H_1 at this order, but also on its derivatives with respect to the slow variable.

A. Derivation

Expansion to order ϵ^4 of the energy E of equation (1) using the two-scale expansion (7a) is carried out in details in the Appendix. After averaging with respect to the fast variable, the result is:

$$\frac{E}{L} = 2\,\epsilon^4 \left(\frac{1}{L\,\epsilon}\int_{-\frac{L\,\epsilon}{2}}^{+\frac{L\,\epsilon}{2}} \mathcal{L}(S)\,\mathrm{d}S\right) + \mathcal{O}(\epsilon^5). \tag{12a}$$

Here, $\mathcal{L}$ is the effective energy density for the envelope H_1

$$\mathcal{L}(S) = -\frac{1}{8}\left(-{H_1}^2(S) + \frac{2-\nu(U)}{16}\,{H_1}^4(S)\right)\cdots \\ + \frac{1}{2}\,{H_1'}^2(S) - \frac{1}{4}\,\frac{\mathrm{d}(H_1(S)\,H_1'(S))}{\mathrm{d}S}, \tag{12b}$$

which depends only on the slow variable S,

$$S = s\,\epsilon.$$

In equation (12b) it is remarkable that the properties of the substrate, be it elastic or fluid, are captured by the single coefficient $\nu(U)$, defined in equation (5). Based on this remark, we define in Section VI a non-linear elastic foundation *equivalent* to a fluid foundation. The reader already familiar with the theory of localized buckling on a nonlinearly elastic foundation, recalled here for the purpose of completeness, can skip directly to this Section VI.

The terms depending on H_1 but not on its derivatives in equation (12b) define an effective potential

$$V_\nu(H_1) = \frac{1}{8}\left(-{H_1}^2 + \frac{2-\nu(U)}{16}\,{H_1}^4\right). \tag{13}$$

In an infinite system $L\,\epsilon \gg 1$, and for buckling patterns such that $H_1' \to 0$ towards the endpoints $S \to \pm\infty$, we can extend to infinity the domain of integration, and get rid of the boundary term:

$$\frac{E}{L} = \frac{2\,\epsilon^4}{L\,\epsilon}\int_{-\infty}^{+\infty}\left[-V_{\nu(U)}(H_1(S)) + \frac{1}{2}\,{H_1'}^2(S)\right]\mathrm{d}S. \tag{14}$$

Note that the condition $H_1' \to 0$ for $S \to \pm\infty$ holds both for the two types of patterns of interest: the extended pattern is such that $H_1'(S) = 0$ everywhere, and the localized patterns such that $H_1(S)$ goes to zero at infinity.

According to the dynamic phase space analogy [30], the integral of the right-hand side of equation (14) can be identified as the action of a particle $H_1(S)$ with unit mass in a potential $V_{\nu(U)}(H_1)$, when time is identified with the slow variable S. Therefore finding the envelope $H_1(S)$ that makes the energy stationary is equivalent to the problem of finding the trajectory of a particle in an effective potential. The equation for the optimal envelope $H_1(S)$ obtained by the Euler-Lagrange equations is just the equation of motion of the equivalent particle:

$$H_1''(S) = -V_{\nu(\mathrm{U})}'(H_1(S)) \tag{15}$$

This non-linear amplitude equation is similar to that derived by Newell and Whitehead [12], and Segel [31] in the context of Rayleigh-Bénard convection.

B. Localized solutions

Localized buckling is described by solutions of equation (15) such that

$$H_1(S) \to 0, \quad H_1'(S) \to 0 \qquad \text{for } S \to \pm\infty. \tag{16}$$

For the equivalent dynamical system [30], this corresponds to a homoclinic orbit to the trivial solution $H = 0$. Given the potential V_ν, defined in equation (13) and plotted in Figure 4a, this homoclinic orbit exists only if

$$2 - \nu(U) > 0. \tag{17}$$

When this condition holds, the homoclinic orbit is that sketched in grey in Figure 4a. The inequality (17) is a condition for the existence of localized buckling solutions. It is always satisfied in the case of a floating Elastica, for which $\nu(U_f) = \nu_f = -2$. The constant 2 in the left-hand side arises out of the geometrically nonlinear terms in the Elastica. It is positive. As a consequence, geometric nonlinearities are sufficient to produce localized buckling,

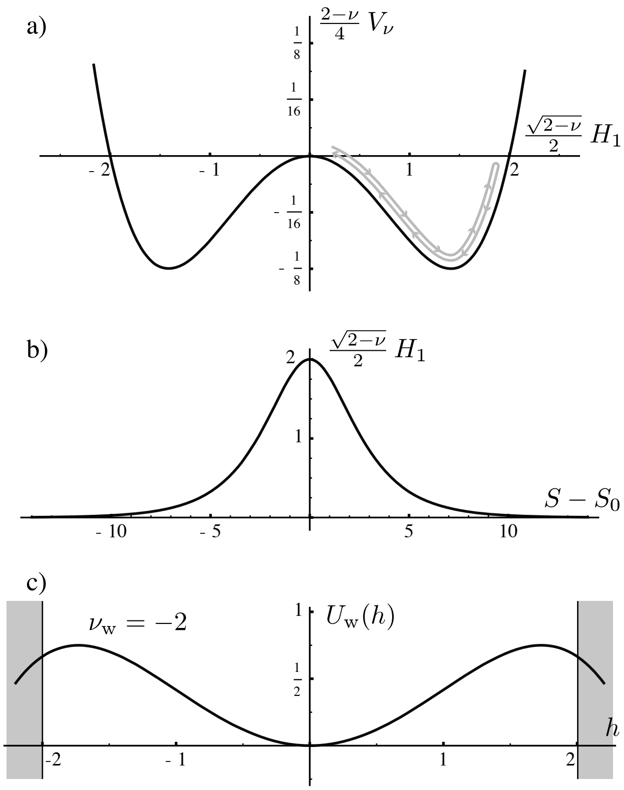

FIG. 4. (a) Effective potential V_ν defined in equation (13), for $2-\nu > 0$. (b) Envelope H_1 describing localized buckling in a floating Elastica. This is the homoclinic trajectory shown by grey path in (a) and given by equation (18). (c) Winkler potential for the equivalent elastic (elastic) foundation. ($\nu_\mathrm{w} = -2$).

even on a linear elastic foundation ($\nu = \nu_\mathrm{w} = 0$), as has been observed by Hunt [25].

Note that the conserved energy $\frac{1}{2}\, {H_1'}^2 + V_{\nu(U)}(H_1)$ associated with the equation of motion (15) takes on the value zero for the homoclinic orbit, by equation (16). This allows one to write the amplitude equation in integrated form:

$$\frac{1}{2}\, {H_1'}^2(S) + V_{\nu(U)}(H_1(S)) = 0.$$

Its solution is a hyperbolic secant:

$$H_1(S) = \frac{4}{(2-\nu(U))^{1/2}}\, \frac{1}{\cosh\left(\frac{S-S_0}{2}\right)}. \tag{18}$$

In terms of the original unknown $h(s)$, the buckling profile reads, using equation (7a):

$$h(s) = \frac{4\,\epsilon}{(2-\nu(U))^{1/2}}\, \frac{\cos s}{\cosh\left(\frac{S-S_0}{2}\right)}. \tag{19}$$

Diamant and Witten [11] study the buckling of a floating Elastica, starting from an Ansatz which is precisely of the form (19). We have just shown that this Ansatz is optimal. This explains the excellent agreement that they obtain when comparing to the numerical and experimental results of Pocivavsek [10].

For future reference, note that by equation (12a), the energy of the localized pattern reads

$$E = \epsilon^3 \left(2 \int_{-\infty}^{+\infty} \mathcal{L}(S)\,\mathrm{d}S\right) = \frac{32\,\epsilon^3}{3\,(2-\nu)} = \frac{8\,\epsilon^3}{3}. \tag{20}$$

VI. EQUIVALENT ELASTIC FOUNDATION

The potential of the foundation $U(h, h')$ comes into the amplitude equation solely through the set of relations (4) and the value of anharmonicity parameter $\nu(U)$. The floating case is characterized by $\nu_\mathrm{f} = 2$ from equation (6), and is therefore equivalent to a non-linear Winkler foundation with parameter

$$\nu_\mathrm{w} = -2. \tag{21}$$

This equivalent elastic foundation is softening ($\nu_\mathrm{w} < 0$), a case which is known to lead to buckling patterns localized far from the boundaries. As noted earlier, this softening, due to the foundation, adds up with that due the geometrical nonlinearities, manifested in the positive constant ($+2$) in the left-hand side of equation (17).

Here we show that the condition (21) defining the equivalent Winkler foundation has a simple interpretation: it makes equal the average value of the dominant anharmonic terms in either the fluid foundation or the equivalent elastic one. To estimate the average strength of the anharmonic term in the fluid case, let us expand U_f,

$$U_\mathrm{f} = \frac{h^2}{2}\left(1 - \frac{{h'}^2}{2} + \cdots\right) = \frac{h^2}{2} - \frac{h^2\,{h'}^2}{4} + \cdots \tag{22}$$

Let then be r the ratio of strength of the anharmonic term in a fluid system, compared to that in a Winkler foundation (3)

$$r = \left(-\frac{\langle h^2\,{h'}^2\rangle}{4}\right) \Big/ \left(\frac{\nu_\mathrm{w}}{24}\,\langle h^4\rangle\right).$$

Here the triangular brackets denote average with respect to the fast variable s. Inserting a harmonic perturbation $h = \epsilon\, H_1\, \cos s$ with critical wavelength $k_\mathrm{c} = 1$ into this expression , we have

$$r = \left(-\frac{1}{4}\,\epsilon^4\, H_1^4\,\langle \cos^2 s\, \sin^2 s\rangle\right) \Big/ \left(\frac{\nu_\mathrm{w}}{24}\,\epsilon^4\, H_1^4\,\langle \cos^4 s\rangle\right)$$

The averages can be calculated by reducing the trigonometric functions: $\langle \cos^2 s\, \sin^2 s\rangle = \frac{1}{8}$ and $\langle \cos^4 s\rangle = \frac{3}{8}$. This yields

$$r = -\frac{2}{\nu_\mathrm{w}}.$$

Therefore, the equivalent Winkler foundation $\nu_\mathrm{w} = -2$ is precisely such that the anharmonic term has the same average intensity as in the fluid foundation U_f, i. e. it follows from the condition $r = 1$.

VII. INTERPRETATION OF THE RESULTS OF POCIVAVSEK ET AL.

Diamant and Witten postulate a buckling profile of the form (19) and show that it reproduces accurately the observations of Pocivavsek [10]. For the sake of completeness, we derive the main results of Diamant and Witten in our own formalism — we refer the reader to their paper for details [11]. The end-to-end shortening Δ of the filament in the localized helix configuration is given by the integral of the factor of P in equation (1b):

$$\Delta = \int 1 - (1 - h'^2)^{\frac{1}{2}}\,\mathrm{d}s = \int \frac{\langle h'^2 \rangle}{2}\,\mathrm{d}s$$
$$= \frac{1}{\epsilon} \int \frac{\epsilon^2\, {H_1}^2 \langle \cos^2 s \rangle}{2}\,\mathrm{d}S = \frac{\epsilon}{4} \int {H_1}^2\,\mathrm{d}S = \frac{16\,\epsilon}{2-\nu}.$$

As a result, the load-displacement relation reads

$$P = 2 - \epsilon^2 = 2 - \left(\frac{2-\nu}{16}\right)^2 \Delta^2 = 2 - \left(\frac{(2-\nu)\,\pi}{8}\right)^2 \left(\frac{\Delta}{\lambda}\right)^2,$$

where we have introduced the wavelength $\lambda = 2\pi$ of the pattern. Restoring dimensional variables and setting $\nu = \nu_\mathrm{f} = 2$, we obtain the load-displacement relation for the localized helix solution:

$$\frac{P}{(B\,K)^{1/2}} = 2 - \frac{\pi^2}{4} \left(\frac{\Delta}{\lambda}\right)^2,$$

where B is the bending modulus of the filament and $K = \rho\,g$ the weight of the fluid per unit volume. Diamant and Witten have noted that this relation accurately matches the numerical findings of Pocivavsek [10].

Note that the energy (20) is consistent with the energy computed by DW in their equation $(13)_\mathrm{DW}$ when we suppress the potential term that is not included in their definition:

$$E_\mathrm{DW} = E - (-P\,\Delta) = \frac{8\,\epsilon^3}{3} + (2 - \epsilon^2)\,\Delta = 2\,\Delta - \frac{\Delta^3}{48}.$$

VIII. CONCLUSION

We have extended the theory of localized buckling of a beam on a non-linear elastic foundation to the case of a floating Elastica. To do so we considered a foundation potential $U(h, h')$ that depends not only on the local deflection h but also on the local slope h'. We considered a general potential $U(h, h')$ satisfying the conditions (4) expressing symmetry assumptions and conventions regarding the zero of energy and the choice of units. We have derived an amplitude equation governing localized buckling patterns in a potential $U(h, h')$, and pointed out an analogy with the standard case of an elastic (Winkler) foundation. The anharmonicity parameter ν_w of the equivalent elastic foundation $U_\mathrm{w}(h)$ matches the parameter $\nu(U)$ of the original potential, which is $\nu_\mathrm{f} = -2$ in the floating case. We have shown that the wrinkle-to-fold transition observed by Pocivavsek is a particular instance of the more general phenomenon of localized buckling, and that the envelope postulated by Diamant and Witten is optimal. This explains the excellent agreement found between the theory based on the Ansatz of Diamant and Witten, with the numerics and experiments of Pocivavsek.

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Appendix: Expansion of energy to fourth order

With the aim to justify equation (12b), we carry out the detailed expansion of energy (1b) to order ϵ^4, using the two-scale expansion (7a) and the critical values (11) of the wavenumber k_c and the load P_c.

Let us first define the energy functional F_ϵ as a function of the envelope $H(S)$. For any function $H(S)$, we consider the associated buckling profile $h(s) = H(s\,\epsilon)\cos s$ as in equation (7a), insert this expression into the density of energy e in equation (1b), average with respect to the fast variable s and finally carry out integration with respect to slow variable $S = \epsilon\, s$. The result is denoted $F_\epsilon(H)$ symbolically, F_ϵ being a functional of the trial form $H(S)$ of the envelope:

$$F_\epsilon(H) = \int \langle e\rangle \,\mathrm{d}S.$$

The explicit dependence of F_ϵ on ϵ comes from the integration with respect to fast variable, and from the parameter $P = P_\mathrm{c} - \epsilon^2$ in the potential energy term.

Let us expand the operator F_ϵ order by order near $H \equiv 0$:

$$F_\epsilon(H) = F^{[0]} + F^{[1]}(H) + \frac{1}{2}\,F^{[2]}(H,H) + \frac{1}{6}\,F^{[3]}(H,H,H) + \cdots \quad \text{(A.1)}$$

where $F^{[i]}(G_1, G_2, \cdots, G_i)$ denote a multilinear, symmetric operator acting on $G_i(S)$ and its derivatives.

Since $h(s)$ depends on the slow variable through a cosine function, any average with respect to the fast variable of the form

$$\langle h^i(s)\, h'^j(s)\, h''^k(s)\rangle$$

is zero when the sum of the integers $i + j + k$ is odd. Since the terms $F^{[1]}(H)$ and $F^{[3]}(H,H,H)$ contain only such averages, those operators vanish:

$$F^{[1]}(H) = 0, \quad F^{[3]}(H,H,H) = 0.$$

Some even terms in the expansion cancel as well. First, the energy e is zero when h is identically 0, by equations (1b) and (4a). As a result, $F^{[0]} = 0$. Second, we note that the operator $F^{[2]}$ yields the energy at order ϵ^2 when the actual envelope $H(S) = \epsilon\, H_1(S)$ is used. This calculation has already been done in Section IV. By identifying with equation (10), we have

$$F^{[2]}(H,H) = \frac{(1 + k_\mathrm{c}^4 - P_\mathrm{c}\, k_\mathrm{c}^2)}{4} \int H^2(S)\,\mathrm{d}S.$$

The critical values (11) of the load P_c and wavenumber k_c imply that the numerical prefactor cancels. As a result, this operator is zero too:

$$F^{[2]}(H,H)=0.$$

We have just shown that the expansion (A.1) starts at order 4 and contains only even powers:

$$F_\epsilon(H)=\frac{1}{24}\,F^{[4]}(H,H,H,H)+\mathcal{O}(|H|^6). \qquad \text{(A.2)}$$

We are interested in calculating the energy based on the two-scale two-scale expansion (7a). Then, H is itself given as an expansion, $H(S)=\epsilon\,H_1(S)+\epsilon^2\,H_2(S)+\cdots$. Inserting this into equation above, we find that the expansion of the energy starts with

$$F_\epsilon(\epsilon\,H_1+\epsilon^2\,H_2+\cdots)=\frac{\epsilon^4}{24}\,F^{[4]}(H_1,H_1,H_1,H_1)+\mathcal{O}(\epsilon^5). \qquad \text{(A.3)}$$

This equation shows that one can neglect all subdominant contributions H_2, H_3 in the calculation of the energy at order ϵ^4. There is no need to keep track of cross-terms like $({H_1}^2\,H_2)$ or $(H_1\,H_3'')$. Even though these terms are formally of order ϵ^4, they cancel out in the end.

We take advantage of this important simplification, and insert

$$h(s)=\epsilon\,H_1(s\,\epsilon)\,\cos s$$

into the energy, instead of the full expansion (7a). Using the notation $f_{[i]}$ for the term of order ϵ^i in the expansion of a quantity f, we have

$$h_{[1]}=H_1(S)\,\cos s \qquad \text{(A.4a)}$$
$$h'_{[1]}=-H_1(S)\,\sin s \qquad \text{(A.4b)}$$
$$h'_{[2]}=H_1'(S)\,\cos s \qquad \text{(A.4c)}$$
$$h''_{[1]}=-H_1(S)\,\cos s \qquad \text{(A.4d)}$$
$$h''_{[2]}=-2\,H_1'(S)\,\sin s \qquad \text{(A.4e)}$$
$$h''_{[3]}=H_1''(S)\,\cos s, \qquad \text{(A.4f)}$$

all other contributions, like $h_{[2]}(s)$ or $h'_{[3]}(s)$, being identically zero.

We are now ready to proceed to the explicit calculation of the energy density e at fourth order in ϵ. Let us expand the first (bending) term in equation (1b), which we denote e_b:

$$e_\mathrm{b}=\frac{h''^2}{2(1-h'^2)}=\frac{h''^2}{2}+\frac{h''^2\,h'^2}{2}+\mathcal{O}(h^6).$$

We average over the fast variable, and extract contribution proportional to ϵ^4:

$$\begin{aligned}\langle e_\mathrm{b}\rangle_{[4]}&=\frac{2\,\langle h''_{[1]}\,h''_{[3]}\rangle+\langle {h''_{[2]}}^2\rangle}{2}+\frac{\langle {h''_{[1]}}^2\,{h'_{[1]}}^2\rangle}{2}\\&=\frac{-2\,H_1\,H_1''\,\langle\cos^2 s\rangle+4\,{H_1'}^2\,\langle\sin^2 s\rangle}{2}+\frac{H_1^4\,\langle\cos^2 s\,\sin^2 s\rangle}{2}\\&=-\frac{1}{2}\,H_1\,H_1''+{H_1'}^2+\frac{1}{16}\,H_1^4. \end{aligned}\qquad \text{(A.5)}$$

A similar calculation yields the potential energy associated with the external compressive force P:

$$e_\mathrm{p}=-P\,(1-\sqrt{1-h'^2})\approx-(2-\epsilon^2)\,\left(\frac{1}{2}h'^2+\frac{1}{8}h'^4\right).$$

The corresponding contribution to order ϵ^4 reads, after averaging with respect to the fast variable,

$$\begin{aligned}\langle e_\mathrm{p}\rangle_{[4]}&=-\langle h'^2\rangle_{[4]}-\frac{1}{4}\,\langle h'^4\rangle_{[4]}+\frac{1}{2}\,\langle h'^2\rangle_{[2]}\\&=-\langle {h'_{[2]}}^2\rangle-\frac{1}{4}\,\langle {h'_{[1]}}^4\rangle+\frac{1}{2}\,\langle {h'_{[1]}}^2\rangle\\&=-\frac{1}{2}\,{H_1'}^2-\frac{3}{32}\,{H_1}^4+\frac{1}{4}\,{H_1}^2. \end{aligned}\qquad \text{(A.6)}$$

Finally, we write a Taylor expansion of the foundation energy $U(h,h')$ to fourth order. We make use of the values of derivatives given in equations (4), and discard all terms that cancel upon averaging with respect to the fast variable — namely all the linear terms, all terms of order 3, and the quartic terms $U_{,h^3h'}(0,0)\,h^3\,h'$ and $U_{,hh'^3}(0,0)\,h\,h'^3$:

$$\begin{aligned}\langle U(h,h')\rangle&=\frac{1}{2}\,\langle h^2\rangle+\frac{1}{24}\,\Big(U_{h^4}(0,0)\,\langle h^4\rangle\cdots\\&\quad+6\,U_{,h^2h'^2}(0,0)\,\langle h^2\,h'^2\rangle+U_{h'^4}(0,0)\,\langle h'^4\rangle\Big).\end{aligned}$$

Extracting the contribution proportional to ϵ^4, we find, by a calculation similar to that done earlier:

$$\langle U(h,h')\rangle_{[4]}=\frac{\nu(U)}{64}\,{H_1}^4, \qquad \text{(A.7)}$$

where $\nu(U)$ is the anharmonicity of the foundation, defined by anticipation in equation (5).

Summing the three contributions to the energy given in equations (A.5), (A.6) and (A.7), we find

$$\langle e\rangle_{[4]}=\frac{1}{4}\,{H_1}^2-\frac{2-\nu}{64}\,{H_1}^4+\frac{1}{2}\,{H_1'}^2-\frac{1}{2}\,H_1\,H_1''. \qquad \text{(A.8)}$$

To prepare for an integration by parts, this is rewritten as

$$\begin{aligned}\langle e\rangle_{[4]}=2\,\Bigg(&-\frac{1}{8}\,\left(-{H_1}^2+\frac{2-\nu}{16}\,{H_1}^4\right)\cdots\\&+\frac{1}{2}\,{H_1'}^2-\frac{1}{4}\,\frac{\mathrm{d}(H_1\,H_1')}{\mathrm{d}S}\Bigg),\end{aligned} \qquad \text{(A.9)}$$

as stated in equations (12a) and (12b).